Nonlocal variables with product-state eigenstates

2001 J. Phys. A: Math. Gen. 34 6881

(http://iopscience.iop.org/0305-4470/34/35/313)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 6881–6889 PII: S0305-4470(01)23354-X

Nonlocal variables with product-state eigenstates

Berry Groisman[1] **and Lev Vaidman**[1,2]

[1] School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel
[2] Centre for Quantum Computation, Department of Physics, University of Oxford, Clarendon Laboratory, Parks Road, Oxford OX1 3PU, UK

Received 22 March 2001
Published 24 August 2001
Online at stacks.iop.org/JPhysA/34/6881

Abstract
An alternative proof for existence of 'quantum nonlocality without entanglement', i.e. existence of variables with product-state eigenstates which cannot be measured locally, is presented. A simple 'nonlocal' variable for the case of one-way communication is given and the limit for its approximate measurability is found.

PACS numbers: 03.65.Ud, 03.65.Ta, 03.67.−a

1. Introduction

A *nonlocal variable* is a property of a compound quantum system which cannot be measured using measurements of local properties only. Aharonov and his collaborators performed an extensive analysis of nonlocal variables [1–3] motivated by the question: to what extent do quantum states and quantum variables have 'physical reality'? Here 'real' corresponds to 'measurable'. For this analysis, it was crucial that the measurements of local properties were performed simultaneously (in some Lorentz frame). The resources were not constrained: measuring devices included, in particular, entangled quantum systems. It was found that there are nonlocal variables which are measurable using only local interactions and prior entanglement. In particular, the Bell operator, the eigenstates of which are four maximally entangled states, is measurable. On the other hand, it was proven that there are unmeasurable variables too.

Today, nonlocal variables have become an important concept for practical applications in the field of quantum communication. The constraint of simultaneity of local measurements is, usually, not relevant for these considerations, but instead there are constraints on resources. The standard question is: what can be measured with local measurements and unlimited classical communication? It is assumed that the measuring devices do not include entangled systems; otherwise, the quantum states of the parts of the composite systems could be all *teleported* [4] to one place and then the 'nonlocal' variable becomes effectively local. The analysis of the locality of variables according to this definition was recently performed by

Bennett *et al* [5] who found that there are variables with product-state eigenstates which are unmeasurable.

In the present work we suggest an alternative, more simple, proof of the main result of Bennett *et al*. We apply our method first to a similar problem in which only one-way classical communication is allowed, and at the end, to a generalization of the result of Bennett *et al* which they suggested as a conjecture.

2. One-way classical communication constraint

We are looking for a variable of a composite system consisting of two parts A and B which has product-state eigenstates $|\Psi_i\rangle = |\phi_i\rangle_a |\psi_i\rangle_b$ and which is not measurable via local interactions and *one-way* communication from A to B. We take a minimal definition of 'measurable': the measurement has to tell with certainty if the system is in a particular eigenstate of the measured variable. There is no requirement that the measurement is ideal, i.e. that the eigenstates are not changed in the process of measurement: it might be a demolition measurement. Given that it identifies all the eigenstates, the linearity of quantum theory ensures that if the initial state is a superposition of the eigenstates, then the measurement will yield the outcomes with the probabilities governed by the quantum theory.

It turns out that there is a very simple example of such a variable. The nondegenerate eigenstates of this variable are:

$$\begin{aligned} |\Psi_1\rangle &= |0\rangle_a \quad |0\rangle_b \\ |\Psi_2\rangle &= |1\rangle_a \quad |0\rangle_b \\ |\Psi_3\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle_a + |1\rangle_a)\, |1\rangle_b \\ |\Psi_4\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle_a - |1\rangle_a) |1\rangle_b \end{aligned} \tag{1}$$

where $|0\rangle_a, |1\rangle_a$ is the basis at A and $|0\rangle_b, |1\rangle_b$ is the basis at B. Let us formulate the problem again. The system is prepared by an external party in one of the four mutually orthogonal product states $|\Psi_i\rangle$. The prepared state is unknown to Alice who is located at A and to Bob who is located at B. The aim of the measurement is to find out in which initial state the system has been prepared. The one-way communication channel is from Alice to Bob. Bob cannot transmit any information to Alice and cannot act on Alice's state; therefore, Alice has to start first. Alice performs a sequence of measurements and local operations on her part of the system and gets a particular outcome k. She can also perform her operations step by step, but there is no principal difference between one step and many steps strategy; 'k' signifies the final outcome after Alice has completed all her measurements. Alice reports outcome k to Bob.

Alice's quantum measurement can be described by two stages: at the first stage the evolution of the quantum state is unitary, at the second stage a collapse of the quantum state (real or effective) occurs. It is enough to consider only the first stage. The unitary evolution on Alice's part can be described as:

$$|\phi_i\rangle_a |A\rangle \mapsto \sum_{k=1}^{K} \alpha_{ik} |w_{ik}\rangle_a \tag{2}$$

where $|A\rangle$ is the initial quantum state of Alice's measuring devices, $|w_{ik}\rangle_a$ is the quantum state of the particle and Alice's measurement devices corresponding to a particular outcome k, and the summation is over all possible outcomes.

There are the following relations between possible initial states in the site of Alice:

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle) \qquad |\phi_4\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle) \tag{3}$$

The unitary evolution (2) keeps these relations:

$$\begin{aligned}\sum_k \alpha_{3k}|w_{3k}\rangle &= \frac{1}{\sqrt{2}}\left(\sum_k \alpha_{1k}|w_{1k}\rangle + \sum_k \alpha_{2k}|w_{2k}\rangle\right)\\ \sum_k \alpha_{4k}|w_{4k}\rangle &= \frac{1}{\sqrt{2}}\left(\sum_k \alpha_{1k}|w_{1k}\rangle - \sum_k \alpha_{2k}|w_{2k}\rangle\right).\end{aligned} \tag{4}$$

We choose amplitudes α_{ik} to be real and nonnegative. This is always possible because the phase can be included in the definition of $|w_{ik}\rangle$. Quantum states $|w_{ik}\rangle$ and $|w_{jk'}\rangle$ with different k and k' are orthogonal because they correspond to different outcomes of the measuring devices (which by definition are macroscopic).Therefore, the relations (4) hold for each k separately:

$$\begin{aligned}\alpha_{3k}|w_{3k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{1k}|w_{1k}\rangle + \alpha_{2k}|w_{2k}\rangle)\\ \alpha_{4k}|w_{4k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{1k}|w_{1k}\rangle - \alpha_{2k}|w_{2k}\rangle).\end{aligned} \tag{5}$$

Initially, the states in Alice's site are mutually orthogonal in each pair: $\langle\phi_1|\phi_2\rangle_a = 0$ and $\langle\phi_3|\phi_4\rangle_a = 0$. Thus, Alice is able to distinguish between the states in each pair. It is important because Bob's corresponding local states are identical, so he cannot distinguish between the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ and between the states $|\Psi_3\rangle$ and $|\Psi_4\rangle$. Therefore, whatever Alice does, she must retain distinguishability between the states in each pair. This means that if a particular outcome k might come out for both initial states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ (or $|\Psi_3\rangle$ and $|\Psi_4\rangle$), then, at every stage, the corresponding quantum states at Alice's site must be orthogonal. This can be formulated in the following equations:

$$\alpha_{1k}\alpha_{2k}\langle w_{1k}|w_{2k}\rangle = 0 \qquad \alpha_{3k}\alpha_{4k}\langle w_{3k}|w_{4k}\rangle = 0. \tag{6}$$

From (5) after some manipulation we obtain:

$$\begin{aligned}\alpha_{1k}^2 &= \frac{1}{2}\big(\alpha_{3k}^2 + \alpha_{4k}^2 + 2\alpha_{3k}\alpha_{4k}\langle w_{3k}|w_{4k}\rangle\big)\\ \alpha_{2k}^2 &= \frac{1}{2}\big(\alpha_{3k}^2 + \alpha_{4k}^2 - 2\alpha_{3k}\alpha_{4k}\langle w_{3k}|w_{4k}\rangle\big)\\ \alpha_{3k}^2 &= \frac{1}{2}\big(\alpha_{1k}^2 + \alpha_{2k}^2 + 2\alpha_{1k}\alpha_{2k}\langle w_{1k}|w_{2k}\rangle\big)\\ \alpha_{4k}^2 &= \frac{1}{2}\big(\alpha_{1k}^2 + \alpha_{2k}^2 - 2\alpha_{1k}\alpha_{2k}\langle w_{1k}|w_{2k}\rangle\big).\end{aligned} \tag{7}$$

Substituting (6) in (7), we obtain four equations for α_{ik} which result in the equality

$$\alpha_{1k} = \alpha_{2k} = \alpha_{3k} = \alpha_{4k}. \tag{8}$$

Therefore, if k is a possible outcome for one initial state $|\Psi_i\rangle$, i.e. the corresponding coefficient α_{ik} does not vanish, then k is a possible outcome for all initial states. From (6) it follows that for all such outcomes k, the orthogonality condition holds:

$$\langle w_{1k}|w_{2k}\rangle = 0. \tag{9}$$

Substituting (8) into (5) we obtain, for each possible outcome k, the following relations:

$$\begin{aligned}|w_{3k}\rangle &= \frac{1}{\sqrt{2}}(|w_{1k}\rangle + |w_{2k}\rangle)\\ |w_{4k}\rangle &= \frac{1}{\sqrt{2}}(|w_{1k}\rangle - |w_{2k}\rangle).\end{aligned} \tag{10}$$

Thus, the evolutions for different initial states of the quantum state of the system and Alice's measuring devices which ended with the outcome k are:

$$\begin{aligned}
|\Psi_1\rangle|A\rangle &\rightarrow \quad |w_{1k}\rangle_a \qquad\qquad |0\rangle_b \\
|\Psi_2\rangle|A\rangle &\rightarrow \quad |w_{2k}\rangle_a \qquad\qquad |0\rangle_b \\
|\Psi_3\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}(|w_{1k}\rangle_a + |w_{2k}\rangle_a)\,|1\rangle_b \\
|\Psi_4\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}(|w_{1k}\rangle_a - |w_{2k}\rangle_a)|1\rangle_b.
\end{aligned} \tag{11}$$

Taking into account (9), we see that this structure is isomorphic with the initial structure (with the correspondence $|0\rangle_a|A\rangle \leftrightarrow |w_{1k}\rangle_a$, $|1\rangle_a|A\rangle \leftrightarrow |w_{2k}\rangle_a)$. Therefore, we have shown that if there is a constraint on Alice's actions such that she cannot lead to a situation in which it is impossible in principle to distinguish with certainty between different initial states $|\Psi_i\rangle$, then she cannot make any progress towards distinguishing the states. Thus, Alice cannot give Bob any useful information. Bob can perform operations on his local part, but he obviously cannot distinguish between the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ and between the states $|\Psi_3\rangle$ and $|\Psi_4\rangle$. This completes the proof of unmeasurability of the variable with nondegenerate eigenstates (1).

Note that this proof is easily generalized for the variable with nondegenerate eigenstates

$$\begin{aligned}
|\Psi_1\rangle &= |0\rangle_a \qquad\qquad\qquad |0\rangle_b \\
|\Psi_2\rangle &= |1\rangle_a \qquad\qquad\qquad |0\rangle_b \\
|\Psi_3\rangle &= (\cos\theta|0\rangle_a + \sin\theta|1\rangle_a)\,|1\rangle_b \\
|\Psi_4\rangle &= (\sin\theta|0\rangle_a - \cos\theta|1\rangle_a)|1\rangle_b
\end{aligned} \tag{12}$$

where $0 < \theta < \frac{\pi}{2}$.

3. The two-way classical communication constraint

In this section we will reproduce the result of Bennett *et al* [5] using the method of the previous section. We will prove that certain variables with product-state eigenstates cannot be measured (even in the above demolition way) using local operations and unlimited classical two-way communication.

The variable which Bennett *et al* found has the following nondegenerate eigenstates:

$$\begin{aligned}
|\Psi_1\rangle &= \tfrac{1}{\sqrt{2}}|0\rangle_a \qquad (|0\rangle_b + |1\rangle_b) \\
|\Psi_2\rangle &= \tfrac{1}{\sqrt{2}}|0\rangle_a \qquad (|0\rangle_b - |1\rangle_b) \\
|\Psi_3\rangle &= \tfrac{1}{\sqrt{2}}|2\rangle_a \qquad (|1\rangle_b + |2\rangle_b) \\
|\Psi_4\rangle &= \tfrac{1}{\sqrt{2}}|2\rangle_a \qquad (|1\rangle_b - |2\rangle_b) \\
|\Psi_5\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle_a + |1\rangle_a)\,|2\rangle_b \\
|\Psi_6\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle_a - |1\rangle_a)|2\rangle_b \\
|\Psi_7\rangle &= \tfrac{1}{\sqrt{2}}(|1\rangle_a + |2\rangle_a)\,|0\rangle_b \\
|\Psi_8\rangle &= \tfrac{1}{\sqrt{2}}(|1\rangle_a - |2\rangle_a)|0\rangle_b \\
|\Psi_9\rangle &= \quad |1\rangle_a \qquad\quad |1\rangle_b
\end{aligned} \tag{13}$$

where $|0\rangle$, $|1\rangle$ and $|2\rangle$ are local bases in Alice's and Bob's sites.

In our approach, in contrast with the original proof, we will show that if we impose the constraint of not allowing for any probability to fail in the measurement, i.e. of reaching a state in which it is in principle impossible to distinguish with certainty between different initial states $|\Psi_i\rangle$, then Alice and Bob cannot make any progress towards completing the measurement. We note that even if the two-way communication is allowed, one party has to start. Since they have only a classical channel, a measurement which ends up with a particular outcome has to be performed in one of the sites. Thus, constructing the proof similar to that of the previous section, but for the variable with the eigenstates (13), is sufficient.

Since the eigenstates (13) have a symmetry between A and B, we can assume without losing generality that the first step is performed by Alice who performs the measurement with possible outcomes k. The unitary evolution on Alice's part can be described as

$$|\phi_i\rangle_a|A\rangle \mapsto \sum_k \alpha_{ik}|w_{ik}\rangle_a. \tag{14}$$

From $|\phi_1\rangle = |\phi_2\rangle$ and $|\phi_3\rangle = |\phi_4\rangle$, we immediately obtain

$$\begin{aligned} &\alpha_{1k} = \alpha_{2k} && \alpha_{3k} = \alpha_{4k} \\ &|w_{1k}\rangle = |w_{2k}\rangle && |w_{3k}\rangle = |w_{4k}\rangle. \end{aligned} \tag{15}$$

The evolution (14) should keep the relations between initial states, and since all states $|w_{ik}\rangle_a$ with different k must be orthogonal, the same relations hold for each individual possible k:

$$\begin{aligned} \alpha_{1k}|w_{1k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{5k}|w_{5k}\rangle + \alpha_{6k}|w_{6k}\rangle) \\ \alpha_{3k}|w_{3k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{7k}|w_{7k}\rangle - \alpha_{8k}|w_{8k}\rangle) \\ \alpha_{5k}|w_{5k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{1k}|w_{1k}\rangle + \alpha_{9k}|w_{9k}\rangle) \\ \alpha_{6k}|w_{6k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{1k}|w_{1k}\rangle - \alpha_{9k}|w_{9k}\rangle) \\ \alpha_{7k}|w_{7k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{9k}|w_{9k}\rangle + \alpha_{3k}|w_{3k}\rangle) \\ \alpha_{8k}|w_{8k}\rangle &= \frac{1}{\sqrt{2}}(\alpha_{9k}|w_{9k}\rangle - \alpha_{3k}|w_{3k}\rangle). \end{aligned} \tag{16}$$

Bob, obviously, cannot distinguish between states $|\Psi_5\rangle$ and $|\Psi_6\rangle$ and between states $|\Psi_7\rangle$ and $|\Psi_8\rangle$. He also might fail to distinguish between $|\Psi_1\rangle$ and $|\Psi_9\rangle$ and between $|\Psi_3\rangle$ and $|\Psi_9\rangle$ because the states $|\Psi_1\rangle$, $|\Psi_3\rangle$ are nonorthogonal to $|\Psi_9\rangle$ on Bob's side. Therefore, Alice should be able to distinguish between these pairs, i.e. for all possible outcomes k the following conditions must be kept at Alice's site:

$$\begin{aligned} &\alpha_{5k}\alpha_{6k}\langle w_{5k}|w_{6k}\rangle = 0 && \alpha_{7k}\alpha_{8k}\langle w_{7k}|w_{8k}\rangle = 0 \\ &\alpha_{1k}\alpha_{9k}\langle w_{1k}|w_{9k}\rangle = 0 && \alpha_{3k}\alpha_{9k}\langle w_{3k}|w_{9k}\rangle = 0. \end{aligned} \tag{17}$$

After some straightforward algebraic manipulations, (16) and (17) yield that all nine coefficients α_{ik} are equal, and that $|w_{1k}\rangle_a$, $|w_{3k}\rangle_a$ and $|w_{9k}\rangle_a$ are mutually orthogonal. Therefore, the evolutions for different initial states of the quantum state of the system and

Alice's measuring devices ended with the outcome k is

$$
\begin{aligned}
|\Psi_1\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}|w_{1k}\rangle_a \qquad (|0\rangle_b + |1\rangle_b)\\
|\Psi_2\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}|w_{1k}\rangle_a \qquad (|0\rangle_b - |1\rangle_b)\\
|\Psi_3\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}|w_{3k}\rangle_a \qquad (|1\rangle_b + |2\rangle_b)\\
|\Psi_4\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}|w_{3k}\rangle_a \qquad (|1\rangle_b - |2\rangle_b)\\
|\Psi_5\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}(|w_{1k}\rangle_a + |w_{9k}\rangle_a)\,|2\rangle_b\\
|\Psi_6\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}(|w_{1k}\rangle_a - |w_{9k}\rangle_a)|2\rangle_b\\
|\Psi_7\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}(|w_{9k}\rangle_a + |w_{3k}\rangle_a)\,|0\rangle_b\\
|\Psi_8\rangle|A\rangle &\rightarrow \tfrac{1}{\sqrt{2}}(|w_{9k}\rangle_a - |w_{3k}\rangle_a)|0\rangle_b\\
|\Psi_9\rangle|A\rangle &\rightarrow \quad |w_{9k}\rangle_a \qquad |1\rangle_b.
\end{aligned}
\tag{18}
$$

This structure is isomorphic with the structure of the initial state (with the correspondence $|0\rangle_a|A\rangle \leftrightarrow |w_{1k}\rangle_a$, $|1\rangle_a|A\rangle \leftrightarrow |w_{9k}\rangle_a$ and $|2\rangle_a|A\rangle \leftrightarrow |w_{3k}\rangle_a$). Therefore, we have shown that if there is a constraint on Alice's actions such that she cannot lead to a situation in which it is impossible in principle to distinguish with certainty between different initial states $|\Psi_i\rangle$, then she cannot make any progress towards distinguishing the states. Thus, Alice cannot give Bob any useful information. If Alice's operation (the first round) yields no progress towards the solution of the problem, then all following rounds cannot change the situation either.

We can apply this method to prove unmeasurability of a more general variable suggested in Bennett's paper as a conjecture. The set of nondegenerate eigenstates of this variable is

$$
\begin{aligned}
|\Psi_1\rangle &= |0\rangle_a \qquad (\cos\eta|0\rangle_b + \sin\eta|1\rangle_b)\\
|\Psi_2\rangle &= |0\rangle_a \qquad (\sin\eta|0\rangle_b - \cos\eta|1\rangle_b)\\
|\Psi_3\rangle &= |2\rangle_a \qquad (\cos\xi|1\rangle_b + \sin\xi|2\rangle_b)\\
|\Psi_4\rangle &= |2\rangle_a \qquad (\sin\xi|1\rangle_b - \cos\xi|1\rangle_b)\\
|\Psi_5\rangle &= (\cos\theta|0\rangle_a + \sin\theta|1\rangle_a)\;|2\rangle_b\\
|\Psi_6\rangle &= (\sin\theta|0\rangle_a - \cos\theta|1\rangle_a)\;|2\rangle_b\\
|\Psi_7\rangle &= (\cos\gamma|1\rangle_a + \sin\gamma|2\rangle_a)\;|0\rangle_b\\
|\Psi_8\rangle &= (\sin\gamma|1\rangle_a - \cos\gamma|2\rangle_a)|0\rangle_b\\
|\Psi_9\rangle &= |1\rangle_a \qquad |1\rangle_b
\end{aligned}
\tag{19}
$$

where all angles $\eta, \xi, \theta, \gamma$ are strictly inside the interval $(0, \frac{\pi}{2})$. Indeed, considering Alice to make the first step and following the arguments above, we obtain again relations (15) and also the following relations:

$$
\begin{aligned}
\alpha_{1k}^2 - \alpha_{9k}^2 &= \cos 2\theta\big(\alpha_{5k}^2 - \alpha_{6k}^2\big)\\
\alpha_{5k}^2 - \alpha_{6k}^2 &= \cos 2\theta\big(\alpha_{1k}^2 - \alpha_{9k}^2\big)\\
\alpha_{9k}^2 - \alpha_{3k}^2 &= \cos 2\gamma\big(\alpha_{7k}^2 - \alpha_{8k}^2\big)\\
\alpha_{7k}^2 - \alpha_{8k}^2 &= \cos 2\gamma\big(\alpha_{9k}^2 - \alpha_{3k}^2\big).
\end{aligned}
\tag{20}
$$

The only solution of these equations is that all coefficients α_{ik} are equal, i.e. that Alice cannot make any progress if she makes the first step. Similar equations are obtained if Bob is to make the first step, so he cannot make any progress either.

4. Optimal local estimation measurements: the case of one-way communication

In the previous sections we proved that 100% reliable measurements of certain variables are impossible. Let us show now, for the case of one-way communication, that not just the ideal case is impossible but also that it is impossible to get close to it. To show this we will relax both the requirement of 100% success and the requirement of 100% reliability, and will ask the following question: what is the optimal measurement which can get the best guess of the prepared state? The 'best' means that on average we obtain maximal probability for the correct guess. If the maximal probability does not approach 1, it means that it is impossible to construct a protocol in which success and reliability will approach 100%.

We consider again the situation after Alice completes the measurements at her site. Equations (2)–(5), (7) still hold, but the orthogonality conditions (6), (9) are not imposed. Alice has to distinguish between the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ and between the states $|\Psi_3\rangle$ and $|\Psi_4\rangle$. For each k she makes her guess according to the maximal coefficient α_{ik} in each pair α_{1k}, α_{2k} and α_{3k}, α_{4k}. Bob distinguishes between the pairs with 100% efficiency; therefore, for a given k, on average according to the initial state, the probability for the correct guess is

$$p = \frac{1}{2}\frac{\max\left\{\alpha_{1k}^2, \alpha_{2k}^2\right\}}{\alpha_{1k}^2 + \alpha_{2k}^2} + \frac{1}{2}\frac{\max\left\{\alpha_{3k}^2, \alpha_{4k}^2\right\}}{\alpha_{3k}^2 + \alpha_{4k}^2}. \tag{21}$$

Our task is to find the strategy for Alice such that, on average on all outcomes k, the probability p will become maximal. As before, since the constraints are separate for each k, we just have to look for a maximum for a particular k. From (7) we obtain

$$\alpha_{1k}^2 + \alpha_{2k}^2 = \alpha_{3k}^2 + \alpha_{4k}^2. \tag{22}$$

Without losing generality we can assume that $\alpha_{1k} > \alpha_{2k}$ and $\alpha_{3k} > \alpha_{4k}$. Let us define parameters γ, ϵ and δ:

$$\alpha_{1k}^2 = \gamma(1+\epsilon) \qquad \alpha_{2k}^2 = \gamma(1-\epsilon)$$
$$\alpha_{3k}^2 = \gamma(1+\delta) \qquad \alpha_{4k}^2 = \gamma(1-\delta).$$

Then p, the probability we have to maximize, becomes

$$p = \frac{2+\epsilon+\delta}{4}. \tag{23}$$

From (7) and (23) we obtain

$$p = \frac{1}{4}\big(2+\epsilon+\sqrt{1-\epsilon^2}\,\langle w_{1k}|w_{2k}\rangle\big). \tag{24}$$

The maximum value of p is obtained when $\langle w_{1k}|w_{2k}\rangle = 1$. Then, the optimization on different values of ϵ yields a maximum for $\epsilon = \frac{1}{\sqrt{2}}$, and thus, the probability for the correct guess of the prepared state is not more than

$$p_{\max} = \frac{1}{2} + \frac{1}{2\sqrt{2}}. \tag{25}$$

This bound is, in fact, tight, since it can be realized via the measurement in the basis

$$\begin{aligned} |\chi_1\rangle_a &= \sin\left(\frac{\pi}{8}\right)|0\rangle_a + \cos\left(\frac{\pi}{8}\right)|1\rangle_a \\ |\chi_2\rangle_a &= \cos\left(\frac{\pi}{8}\right)|0\rangle_a - \sin\left(\frac{\pi}{8}\right)|1\rangle_a. \end{aligned} \tag{26}$$

If the state $|\chi_1\rangle$ is obtained, then the team announces that its guess is $|\Psi_2\rangle$ or $|\Psi_3\rangle$, according to the results of Bob, $|0\rangle_b$ or $|1\rangle_b$. If the state $|\chi_2\rangle$ is obtained, then the team announces $|\Psi_1\rangle$

or $|\Psi_4\rangle$ according to the results of Bob. It can be seen from a straightforward trigonometric manipulation that the probability for the correct result is indeed $p_{\max}$:

$$|\langle\phi_2|\chi_1\rangle|^2 = |\langle\phi_3|\chi_1\rangle|^2 = |\langle\phi_1|\chi_2\rangle|^2 = |\langle\phi_4|\chi_2\rangle|^2 = \frac{1}{2} + \frac{1}{2\sqrt{2}}. \tag{27}$$

Note that the probability of the successful guess is the same for all initial states $|\Psi_i\rangle$.

5. Conclusions

In this paper we have discussed measurability of variables of a composite system consisting of two separated parts. The variables we have considered have nondegenerate product-state eigenstates. We have shown that, assuming one-way classical communication and local interactions, there is a simple example of an unmeasurable variable of this kind with eigenstates given by (1). We have simplified the proof and made certain generalizations of the results by Bennett *et al* regarding measurability of such variables when two-way classical communication is allowed.

It was shown before [3] that the variable with product-state eigenstates (1) is unmeasurable. However, this proof was under different assumptions. The main difference is that the proof was only for unmeasurability of ideal (nondemolition) measurements. In this case it was easy to show that measurability leads to superluminal signalling and this was the proof that it is impossible. The current work considers the more difficult question of the possibility of 'demolition' measurements in which it is not required that the eigenstates are unchanged during the measurement. Under certain constraints regarding allowed operations, it has been shown that a variable with an entangled (but not maximally entangled) eigenstate cannot be measured even in a demolition way [1, 2]. When the constraints were removed [3], the unmeasurability was not true anymore, but it was shown that measurement of such a variable invariably erases relevant local information. It seems that more results can be obtained in this framework. In particular, a recently developed formalism of *semicausal* operations seems very promising as it has already led to useful results in quantum communication [6].

Just before the completion of this work, an actual experiment with the eigenstates (13) has been proposed by Carollo *et al* [7]. An experiment with nine eigenstates (13) is significantly more difficult than an experiment with four states (1). The impossibility of distinguishing the states (1) with local measurements and one-way classical communication represents the same basic feature as the impossibility of distinguishing the states (13) with the two-way communication channel. Thus, we suggest modifying this experimental proposal for the case of four states and to start with this easier experiment.

Carollo *et al* also proved that even if the nine eigenstates are available locally, they cannot be discriminated using linear optical elements. It means that even the existence of prior entanglement does not allow reliable discrimination of the nine bipartite states (13) with linear elements, and not just because the teleportation cannot be performed [8, 9]. The question of reliable discrimination of the four bipartite states (1) with linear optics, one-way communication and prior entanglement remains open.

Acknowledgments

It is a pleasure to thank Noam Erez, Christopher Fuchs, Lior Goldenberg, Tal Mor, Christoph Simon, Stephen Wiesner, and Nadav Yoran for helpful discussions. This research was supported in part by grant 471/98 of the Basic Research Foundation (administered by the Israel Academy of Sciences and Humanities) and the EPSRC grant GR/N33058.

References

[1] Aharonov Y and Albert D 1981 *Phys. Rev.* D **24** 359
[2] Aharonov Y, Albert D and Vaidman L 1986 *Phys. Rev.* D **34** 1805
[3] Popescu S and Vaidman L 1994 *Phys. Rev.* A **49** 4331
[4] Bennett C H, Brassard G, Grepeau C, Jozsa R, Peres A and Wootters W 1993 *Phys. Rev. Lett.* **70** 1895
[5] Bennett C H, DiVincenzo D P, Fuchs C A, Mor T, Rains E, Shor P W, Smolin J A and Wootters W K 1999 *Phys. Rev.* A **59** 1070
[6] Beckman D, Gottesman D, Nielsen M A and Preskill J 2001 *Preprint* quant-ph/0102043
[7] Carollo A, Palma G M, Simon C and Zeilinger A 2001 *Preprint* quant-ph/0102124 (*Phys. Rev.* A at press)
[8] Vaidman L and Yoran N 1999 *Phys. Rev.* A **59** 116
[9] Lütkenhaus N, Calsamiglia J and Suominen K-A 1999 *Phys. Rev.* A **59** 3295